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Wave breaking for a modified two-component Camassa–Holm system

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ABSTRACT

In this paper, we establish sufficient conditions on the initial data to guarantee blow-up phenomenon for the modified two-component Camassa–Holm (MCH2) system.

1. Introduction

We consider the following modified two-component Camassa–Holm system

$$\begin{cases} y_t + u y_x + 2 y u_x = -g \rho \bar{\rho}_x, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ y(0,x) = y_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R} \end{cases} \tag{1.1}$$

where $y = (1 - \partial_x^2)u$, $\rho = (1 - \partial_x^2)(\bar{\rho} - \bar{\rho}_0)$, u denotes the velocity field and $\bar{\rho}_0$ is taken to be a constant, g is the downward constant acceleration of gravity in applications to shallow water waves, for convenience, we let $g = 1$ in this paper.

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Let $\Lambda = (1 - \partial_x^2)^{1/2}$, then the operator Λ^{-2} can be expressed by it's associated Green's function $G(x) = e^{-|x|}/2$ with

$$\Lambda^{-2} f(x) = G * f(x) = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-y|} f(y)\,dy.$$

Let $\gamma = \bar{\rho} - \bar{\rho}_0$, then $G * \rho = \gamma$. So system (1.1) is equivalent to the following one

$$\begin{cases} u_t + uu_x + \partial_x\left(G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right)\right) = 0, & t > 0,\ x \in \mathbb{R}, \\ \gamma_t + u\gamma_x + G * \left((u_x\gamma_x)_x + u_x\gamma\right) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \gamma(0,x) = \gamma_0(x), & x \in \mathbb{R}. \end{cases}$$

This MCH2 system does admit peaked solutions in the velocity and average density, we refer this to Ref. [25] for details. Some other recent work can be found in [18,22]. We find that the MCH2 system is expressed in terms of an averaged or filtered density $\bar{\rho}$ in analogy to the relation between momentum and velocity by setting $\rho = (1 - \partial_x^2)(\bar{\rho} - \bar{\rho}_0)$, but it may not be integrable unlike the following two-component Camassa–Holm (CH2) system.

$$\begin{cases} u_t + uu_x + \partial_x\left(G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{\sigma}{2}\rho^2\right)\right) = 0, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}. \end{cases}$$

The CH2 system appears initially in [33], and recently Constantin and Ivanov in [10] gave a demonstration about its derivation in view of the fluid shallow water theory from the hydrodynamic point of view. This generalization, similarly to the Camassa–Holm equation, possessed the peakon, multi-kink solutions and the bi-Hamiltonian structure [4,16] and is integrable. Well-posedness and wave breaking mechanism were discussed in [14,19,20] and the existence of global solutions was analyzed in [10,19,21]. The geometric investigation can be found in [15,26]. It is not difficult to find that the MCH2 system is a modified version of the CH2 system to allow a dependence on the average density $\bar{\rho}$ (or depth, in the shallow water interpretation) as well as the pointwise density ρ. The characteristic is that it will amount to strengthening the norm for $\bar{\rho}$ from L^2 to H^1 in the potential energy term [22]. It means the following conserved quantity

$$\int_{\mathbb{R}} \left(u^2 + u_x^2 + \gamma^2 + \gamma_x^2\right) dx.$$

Note that for the CH2 system, we cannot obtain the conservation of H^1 norm. This makes our deep research possible and interesting for the MCH2 system.

Obviously, under the constraint of $\rho(x,t) = 0$, system (1.1) reduces to the celebrated Camassa–Holm equation, which was derived physically by Camassa and Holm in [3] (found earlier by Fokas and Fuchssteiner [17] as a bi-Hamiltonian generalization of the KdV equation) by directly approximating the Hamiltonian for Euler's equation in the shallow water region with $u(x,t)$ representing the free surface above a flat bottom. The alternative derivations were provided by [27,28]. The Camassa–Holm equation is completely integrable [11,12] and has infinitely many conservation laws [30]. Local well-posedness for the initial datum $u_0(x) \in H^s$ with $s > 3/2$ was proved in [8,31]. One of the remarkable features of the Camassa–Holm equation is the presence of breaking waves and global solutions in time. Wave breaking for a large class of initial data has been established in [8,9,31,32,34,35], global solutions were also explored in [8,9]. The solution to the Camassa–Holm equation can be continued uniquely after wave breaking either as a conservative global solution or as a dissipative global solution [1,2]. The solitary waves of the Camassa–Holm equation are peaked solitons. The peakons replicate a feature that is characteristic for the waves of great height–waves of largest amplitude that are exact solutions of the governing equations for irrotational water waves [5,7]. The orbital stability of the peakons was shown by Constantin and Strauss in [13]. The property of propagation speed of solutions to the Camassa–Holm equation was discussed in [6,23] while Zhou and his collaborators [24] offered an in-depth study.

This paper is dedicated to the study on wave breaking phenomenon, with aim at the comparison to the blow-up for two-component Camassa–Holm system. It is organized as follows. In Section 2, we will present some results obtained by others, which will be used in this paper. In Section 3, some new blow-up criteria are presented.

2. Preliminaries

In this section, we recall some elementary results. For completeness, we list them and skip their proofs for conciseness. Local well-posedness for (1.1) can be obtained by Kato's semigroup theory [29]. In [18], the authors gave a detailed description on well-posedness theorem.

Theorem 2.1. *(See [18].) Given $X_0 = (u_0, \gamma_0)^T \in H^s \times H^{s-1}$, $s \geqslant 5/2$, there exist a maximal $T = T(\|X_0\|_{H^s \times H^{s-1}}) > 0$, and a unique solution $X_0 = (u, \gamma)^T$ to system (1.1) such that*

$$X = X(\cdot, X_0) \in C\big([0, T); H^s \times H^{s-1}\big) \cap C^1\big([0, T); H^{s-1} \times H^{s-2}\big).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$X_0 \to X(\cdot, X_0) : H^s \times H^s \to C\big([0, T); H^s \times H^{s-1}\big) \cap C^1\big([0, T); H^{s-1} \times H^{s-2}\big)$$

is continuous.

Next result describes the precise blow-up scenarios for sufficiently regular solutions to our system.

Lemma 2.2. *(See [18].) Let $X_0 = (u_0, \gamma_0)^T \in H^s \times H^{s-1}$, $s \geqslant 5/2$, and let T be the maximal existence time of the solution $X = (u, \gamma)^T$ to system (1.1) with the initial X_0. Then the corresponding solution blows up in finite time if and only if*

$$\lim_{t \to T} \inf_{x \in \mathbb{R}} \big\{u_x(x, t)\big\} = -\infty.$$

Lemma 2.3. *(See [36].) Assume that a differentiable function $y(t)$ satisfies*

$$y'(t) \leqslant -Cy^2(t) + K \tag{2.1}$$

with constants $C, K > 0$. If the initial datum $y(0) = y_0 < -\sqrt{\frac{K}{C}}$, then the solution to (2.1) goes to $-\infty$ before t tend to $\frac{1}{-Cy_0 + \frac{K}{y_0}}$.

Lemma 2.4. *(See [37].) Suppose that $\Psi(t)$ is twice continuously differential satisfying*

$$\begin{cases} \Psi''(t) \geqslant C_0 \Psi'(t) \Psi(t), & t > 0,\ C_0 > 0, \\ \Psi(t) > 0, & \Psi'(t) > 0. \end{cases} \tag{2.2}$$

Then $\psi(t)$ blows up in finite time. Moreover the blow-up time can be estimated in terms of the initial datum as

$$T \leqslant \max\left\{\frac{2}{C_0\Psi(0)}, \frac{\Psi(0)}{\Psi'(0)}\right\}.$$

We also need to introduce the standard particle trajectory method for later use. Suppose $u(x,t)$ solves (1.1), $q(x,t)$ satisfies the following initial value problem:

$$\begin{cases} q_t = u(q,t), & 0 < t < T,\ x \in \mathbb{R}, \\ q(x,0) = x, & x \in \mathbb{R}, \end{cases}$$

where T is the life span of the solution, then q is a diffeomorphism of the line. Differentiating the above equation with respect to x, one has

$$\frac{dq_t}{dx} = q_{xt} = u_x(q,t)q_x, \quad t \in (0,T).$$

Hence

$$q_x(x,t) = \exp\left\{\int_0^t u_x(q,s)\,ds\right\}, \quad q_x(x,0) = 1. \tag{2.3}$$

Thus, we know the map $q(\cdot,t)$ is an increasing diffeomorphism of $\mathbb{R}$.

3. Blow-up

In this section, we establish sufficient conditions on the initial data to guarantee blow-up for system (1.1).

Theorem 3.1. *Suppose $X_0 = (u_0, \gamma_0)^T \in H^s \times H^{s-1}$, $s \geqslant 5/2$, $\rho_0(x_0) = y_0(x_0) = 0$, and the initial data satisfies the following conditions*:

(i) $\rho_0(x) \geqslant 0$ *on* $(-\infty, x_0)$ *and* $\rho_0(x) \leqslant 0$ *on* (x_0, ∞);
(ii) $\int_{-\infty}^{x_0} e^{\xi} y_0(\xi)\,d\xi \geqslant 0$ *and* $\int_{x_0}^{\infty} e^{-\xi} y_0(\xi)\,d\xi \leqslant 0$,

for some point $x_0 \in \mathbb{R}$. Then the solution to our system (1.1) *with the initial value X_0 blows up in finite time.*

Proof. Differentiating the first equation in the equivalent system with respect to variable x, we obtain

$$u_{tx} + uu_{xx} + u_x^2 + \partial_x^2\left(G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right)\right) = 0.$$

Applying $\partial_x^2(G * f) = G * f - f$ to the above equation yields

$$u_{tx} + uu_{xx} = u^2 - \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2 - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right).$$

This equation gives

$$\begin{aligned}\frac{d}{dt}u_x\big(q(x_0,t),t\big) &= (u_{xt}+uu_{xx})\big(q(x_0,t),t\big)\\ &\leqslant \frac{1}{2}u^2\big(q(x_0,t),t\big)-\frac{1}{2}u_x^2\big(q(x_0,t),t\big)\\ &\quad+\frac{1}{2}\gamma^2\big(q(x_0,t),t\big)-\frac{1}{2}\gamma_x^2\big(q(x_0,t),t\big)-\frac{1}{2}G*\big(\gamma^2-\gamma_x^2\big),\end{aligned}\tag{3.1}$$

where we used the fact

$$G*\left(u^2+\frac{1}{2}u_x^2\right)\geqslant\frac{1}{2}u^2(x).$$

In order to arrive at our result, we need the following three claims.

Claim 1. *$y(q(x_0,t),t)=0$ for all t in its lifespan.*

It is worth noting the equivalent form of the first equation in (1.1) in what follows

$$y_t+2yu_x+y_xu+\rho\gamma_x=0.$$

Applying particle trajectory method to the above equation and the second equation in (1.1), we obtain

$$\begin{aligned}\frac{d}{dt}\big(y\big(q(x,t),t\big)q_x^2(x,t)\big) &= (y_t+2yu_x+y_xu)\big(q(x,t),t\big)q_x^2(x,t)\\ &= -\rho\big(q(x,t),t\big)\gamma_x\big(q(x,t),t\big)q_x^2(x,t),\end{aligned}$$

and

$$\frac{d}{dt}\big(\rho\big(q(x,t),t\big)q_x(x,t)\big)=0,$$

which implies that

$$\rho\big(q(x,t),t\big)q_x(x,t)=\rho_0(x).$$

Obviously, $\rho(q(x_0,t),t)=0$ since $\rho_0(x_0)=0$ and

$$\frac{d}{dt}\big(y\big(q(x_0,t),t\big)q_x^2(x_0,t)\big)=0.$$

Thus $y(q(x_0,t),t)q_x^2(x_0,t)$ is independent of time t. We get by taking $t=0$ without loss of generality

$$y\big(q(x_0,t),t\big)q_x^2(x_0,t)=y_0(x_0)=0.$$

Therefore, thanks to (2.3) we obtain $y(q(x_0,t),t)=0$ for all t in its lifespan.

Claim 2. *For any fixed t, $\gamma_x^2(x,t)-\gamma^2(x,t)\leqslant(\gamma_x^2-\gamma^2)(q(x_0,t),t)$ for all $x\in\mathbb{R}$.*

For any fixed t, if $x \leqslant q(x_0,t)$, then

$$\begin{aligned}
\gamma_x^2(x,t)-\gamma^2(x,t) &= -\left(\int_{-\infty}^{q(x_0,t)} e^{\xi}\rho(\xi,t)\,d\xi - \int_{x}^{q(x_0,t)} e^{\xi}\rho(\xi,t)\,d\xi\right)\\
&\quad\times\left(\int_{q(x_0,t)}^{\infty} e^{-\xi}\rho(\xi,t)\,d\xi + \int_{x}^{q(x_0,t)} e^{-\xi}\rho(\xi,t)\,d\xi\right)\\
&= \gamma_x^2\big(q(x_0,t),t\big)-\gamma^2\big(q(x_0,t),t\big)\\
&\quad - \int_{-\infty}^{x} e^{\xi}\rho(\xi,t)\,d\xi \int_{x}^{q(x_0,t)} e^{-\xi}\rho(\xi,t)\,d\xi\\
&\quad + \int_{x}^{q(x_0,t)} e^{\xi}\rho(\xi,t)\,d\xi \int_{q(x_0,t)}^{\infty} e^{-\xi}\rho(\xi,t)\,d\xi\\
&\leqslant \gamma_x^2\big(q(x_0,t),t\big)-\gamma^2\big(q(x_0,t),t\big), \qquad (3.2)
\end{aligned}$$

where the condition (i) is used. Similarly, for $x \geqslant q(x_0,t)$, we also have

$$\gamma_x^2(x,t)-\gamma^2(x,t) \leqslant \gamma_x^2\big(q(x_0,t),t\big)-\gamma^2\big(q(x_0,t),t\big). \qquad (3.3)$$

Combining (3.2) and (3.3) together, we get that for any fixed t,

$$\gamma_x^2(x,t)-\gamma^2(x,t) \leqslant \gamma_x^2\big(q(x_0,t),t\big)-\gamma^2\big(q(x_0,t),t\big),$$

for all $x \in \mathbb{R}$.

Combining Claim 2 with (3.1), we get

$$\frac{d}{dt}u_x\big(q(x_0,t),t\big) \leqslant \frac{1}{2}u^2\big(q(x_0,t),t\big)-\frac{1}{2}u_x^2\big(q(x_0,t),t\big). \qquad (3.4)$$

Claim 3. *$u_x(q(x_0,t),t)<0$ is decreasing, $u^2(q(x_0,t),t)<u_x^2(q(x_0,t),t)$ for all $t \geqslant 0$.*

Suppose not, i.e., there exists a t_0 such that $u^2(q(x_0,t),t)<u_x^2(q(x_0,t),t)$ on $[0,t_0)$ and $u^2(q(x_0,t_0),t_0)=u_x^2(q(x_0,t_0),t_0)$. Now, let

$$I(t) := \frac{1}{2}e^{-q(x_0,t)} \int_{-\infty}^{q(x_0,t)} e^{\xi}y(\xi,t)\,d\xi$$

and

$$II(t) := \frac{1}{2}e^{q(x_0,t)} \int_{q(x_0,t)}^{\infty} e^{-\xi}y(\xi,t)\,d\xi.$$

Firstly, differentiating $I(t)$, we have

$$\begin{aligned}
\frac{dI(t)}{dt} &= -\frac{1}{2}u\big(q(x_0,t),t\big)e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y(\xi,t)\,d\xi + \frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y_t(\xi,t)\,d\xi \\
&= \frac{1}{2}u(u_x-u)\big(q(x_0,t),t\big) - \frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}(uy_x+2u_xy+\rho\gamma_x)\,d\xi \\
&\geqslant \frac{1}{2}u(u_x-u)\big(q(x_0,t),t\big) + \frac{1}{4}\big(u^2+u_x^2-2uu_x\big)\big(q(x_0,t),t\big) \\
&\quad - \frac{1}{4}\gamma^2\big(q(x_0,t),t\big) + \frac{1}{4}\gamma_x^2\big(q(x_0,t),t\big) + G*\left(\frac{1}{4}\gamma^2-\frac{1}{4}\gamma_x^2\right), \\
&\geqslant \frac{1}{4}\big(u_x^2-u^2\big)\big(q(x_0,t),t\big) - \frac{1}{4}\gamma^2\big(q(x_0,t),t\big) + \frac{1}{4}\gamma_x^2\big(q(x_0,t),t\big) \\
&\quad + G*\left(\frac{1}{4}\gamma^2-\frac{1}{4}\gamma_x^2\right), \\
&= \frac{1}{4}u_x^2\big(q(x_0,t),t\big) - \frac{1}{4}u^2\big(q(x_0,t),t\big) > 0, \quad \text{on } [0,t_0),
\end{aligned} \tag{3.5}$$

where we used Claim 2. Secondly, by the same argument, we get

$$\begin{aligned}
\frac{dII(t)}{dt} &= \frac{1}{2}u\big(q(x_0,t),t\big)e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y(\xi,t)\,d\xi + \frac{1}{2}e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y_t(\xi,t)\,d\xi \\
&= \frac{1}{2}u(u_x+u)\big(q(x_0,t),t\big) - \frac{1}{2}e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}(uy_x+2u_xy+\rho\gamma_x)\,d\xi \\
&\leqslant \frac{1}{2}u(u_x+u)\big(q(x_0,t),t\big) - \frac{1}{4}\big(u^2+u_x^2+2uu_x\big)\big(q(x_0,t),t\big) \\
&\quad + \frac{1}{4}\gamma^2\big(q(x_0,t),t\big) - \frac{1}{4}\gamma_x^2\big(q(x_0,t),t\big) - G*\left(\frac{1}{4}\gamma^2-\frac{1}{4}\gamma_x^2\right), \\
&\leqslant -\frac{1}{4}\big(u_x^2-u^2\big)\big(q(x_0,t),t\big) + \frac{1}{4}\gamma^2\big(q(x_0,t),t\big) - \frac{1}{4}\gamma_x^2\big(q(x_0,t),t\big) \\
&\quad - G*\left(\frac{1}{4}\gamma^2-\frac{1}{4}\gamma_x^2\right), \\
&= -\frac{1}{4}u_x^2\big(q(x_0,t),t\big) + \frac{1}{4}u^2\big(q(x_0,t),t\big) < 0, \quad \text{on } [0,t_0).
\end{aligned} \tag{3.6}$$

Hence, it follows from (3.5), (3.6) and the continuity property of ODEs that

$$u_x^2\big(q(x_0,t),t\big) - u^2\big(q(x_0,t),t\big) = -4I(t)II(t) > -4I(0)II(0) \geqslant 0,$$

for all $t > 0$, where we have used the condition (ii). This implies t_0 can be extend to the infinity.

Using (3.5) and (3.6) again, we have the following equation for $(u_x^2-u^2)(q(x_0,t),t)$:

$$\begin{aligned}\frac{d}{dt}\big(u_x^2-u^2\big)\big(q(x_0,t),t\big) &= -4\frac{d}{dt}I(t)II(t)\\ &= -4II(t)\frac{d}{dt}I(t)-4I(t)\frac{d}{dt}II(t)\\ &\geqslant -II(t)\big(u_x^2-u^2\big)\big(q(x_0,t),t\big)+I(t)\big(u_x^2-u^2\big)\big(q(x_0,t),t\big)\\ &= -u_x\big(q(x_0,t),t\big)\big(u_x^2-u^2\big)\big(q(x_0,t),t\big), \end{aligned}\tag{3.7}$$

where we used $u_x(q(x_0,t),t)=-I(t)+II(t)$.

Now, substituting (3.4) into (3.7), it yields

$$\begin{aligned}\frac{d}{dt}\big(u_x^2-u^2\big)\big(q(x_0,t),t\big) \geqslant{} & \frac{1}{2}\big(u_x^2-u^2\big)\big(q(x_0,t),t\big)\\ &\times\left(\int_0^t\big(u_x^2-u^2\big)\big(q(x_0,t),t\big)\,d\tau-2u_{0x}(x_0)\right).\end{aligned}\tag{3.8}$$

Let $\Psi(t)=\int_0^t(u_x^2-u^2)(q(x_0,t),t)d\tau-2u_{0x}(x_0)$, then (3.8) is an equation of type (2.2) with $C_0=1/2$. The proof is completed by applying Lemma 2.4. $\square$

Remark 3.1. Scrutinizing the proof, we find that the condition (i) actually guarantees the sign of two terms in (3.2) to make the inequality hold. Therefore, it can be replaced by

$$\rho_0(x)\leqslant 0 \quad \text{on } (-\infty,x_0) \quad \text{and} \quad \rho_0(x)\geqslant 0 \quad \text{on } (x_0,\infty),$$

the theorem still holds. On the other hand, it is well known that McKean [32] states that it is the sign of the initial potential $y_0(x)$ not the size of it affects wave breaking phenomenon. Our result generalizes the one into two components case, and shows that the sign of the initial density $\rho_0(x)$ also plays an important role in wave breaking. Note that the result in [22] is different from the current one. The method used here is to make comparison with the previous results for Camassa–Holm equation by McKean. Furthermore, to our knowledge, it is the first time to combine the initial density and potential to establish blow-up result for the MCH2 system.

Theorem 3.2. *Suppose $X_0=(u_0,\gamma_0)^T\in H^s\times H^{s-1}$, $s>5/2$, ρ_0 and u_0 satisfy the following conditions*:

(i) $\rho_0\geqslant 0$ *on* $(-\infty,x_0)$ *and* $\rho_0\leqslant 0$ *on* (x_0,∞) (*or* $\rho_0\leqslant 0$ *on* $(-\infty,x_0)$ *and* $\rho_0\geqslant 0$ *on* (x_0,∞)).
(ii) $u_0'(x_0)\leqslant -\frac{\sqrt{2}}{2}(\|u_0\|_{H^1}^2+\|\gamma_0\|_{H^1}^2)^{1/2}$

for some point $x_0\in\mathbb{R}$. Then the solution to our system (1.1) *with the initial value X_0 blows up in finite time.*

Proof. As mentioned in Claim 2 of Theorem 3.1, condition (i) means that for any fixed t, $\gamma_x^2(x,t)-\gamma^2(x,t)\leqslant(\gamma_x^2-\gamma^2)(q(x_0,t),t)$ for all $x\in\mathbb{R}$. Then

$$\frac{d}{dt}u_x\big(q(x_0,t),t\big)\leqslant\frac{1}{2}u^2\big(q(x_0,t),t\big)-\frac{1}{2}u_x^2\big(q(x_0,t),t\big)$$

$$\leqslant -\frac{1}{2}u_x^2\big(q(x_0,t),t\big)+\frac{1}{4}\|u\|_{H^1}^2$$
$$\leqslant -\frac{1}{2}u_x^2\big(q(x_0,t),t\big)+\frac{1}{4}\big(\|u_0\|_{H^1}^2+\|\gamma_0\|_{H^1}^2\big),$$

setting $\varphi(t)=u_x(q(x_0,t),t)$, we obtain

$$\frac{d\varphi}{dt}=-\frac{1}{2}\varphi^2+K^2,$$

where $K=\frac{1}{2}(\|u_0\|_{H^1}^2+\|\gamma_0\|_{H^1}^2)^{\frac{1}{2}}$.

By applying Lemma 2.3, we have

$$\lim_{t\to T}\varphi(t)=-\infty \quad \text{with } T=\frac{1}{-\frac{1}{2}\varphi_0-\frac{K^2}{\varphi_0}},$$

provided that

$$\varphi_0<-\sqrt{2}K=-\frac{\sqrt{2}}{2}\big(\|u_0\|_{H^1}^2+\|\gamma_0\|_{H^1}^2\big)^{1/2}.$$

This completes the proof. □

Theorem 3.3. *Suppose $X_0=(u_0,\gamma_0)^T\in H^s\times H^{s-1}$, $s>5/2$, u_0 and ρ_0 satisfy the following conditions*:

(i) $\rho_0\geqslant 0$ *on* $(-\infty,0)$ *and* $\rho_0\leqslant 0$ *on* $(0,\infty)$
(*or* $\rho_0\leqslant 0$ *on* $(-\infty,0)$ *and* $\rho_0\geqslant 0$ *on* $(0,\infty)$).
(ii) u_0 *is odd, furthermore* $u_0'(0)<0$.

Then the solution to our system (1.1) *with the initial value X_0 blows up in finite time.*

Proof. Since u_0 is odd, then u is also odd and

$$u(0,t)=u_{xx}(0,t)=0.$$

As mentioned in Claim 2 of Theorem 3.1, condition (i) means that for any fixed t, $\gamma_x^2(x,t)-\gamma^2(x,t)\leqslant(\gamma_x^2-\gamma^2)(q(0,t),t)$ for all $x\in\mathbb{R}$.

So

$$\frac{d}{dt}u_x\big(q(0,t),t\big)\leqslant -\frac{1}{2}u_x^2\big(q(0,t),t\big)$$

setting $\varphi(t)=u_x(q(0,t),t)$, we obtain

$$\frac{d\varphi}{dt}=-\frac{1}{2}\varphi^2.$$

By applying Lemma 2.3, we have

$$\lim_{t\to T}\varphi(t)=-\infty \quad \text{with } T=\frac{1}{-\frac{1}{2}\varphi_0},$$

provided that

$$\varphi_0 < 0.$$

This completes the proof. $\square$

Remark 3.2. The condition of Theorem 4.2 in [18] requires that u_0 and γ_0 are odd, furthermore

$$u_0'(x_0) \leqslant -\frac{\sqrt{2}}{2}\left(\|u_0\|_{H^1}^2 + \|\gamma_0\|_{H^1}^2\right)^{\frac{1}{2}}.$$

Here, we add the condition (i) on the profile of ρ and improve the condition on the slope of u.

Theorem 3.4. *Suppose $X_0 = (u_0, \gamma_0)^T \in H^s \times H^{s-1}$, $s > 5/2$, y_0 and ρ_0 satisfy the following conditions*:

(i) $\rho_0 \geqslant 0$ *on* $(-\infty, 0)$ *and* $\rho_0 \leqslant 0$ *on* $(0, \infty)$
(*or* $\rho_0 \leqslant 0$ *on* $(-\infty, 0)$ *and* $\rho_0 \geqslant 0$ *on* $(0, \infty)$).
(ii) y_0 *is odd, furthermore* $\int_0^\infty e^{-\xi} y_0(\xi)\, d\xi < 0$.

Then the solution to our system (1.1) *with the initial value* X_0 *blows up in finite time.*

Proof. If y_0 is odd, then

$$\begin{aligned} u_0(x) = G * y_0(x) &= \frac{1}{2}\int_{\mathbf{R}} e^{-|x-\xi|} y_0(\xi)\, d\xi \\ &= \frac{1}{2}\int_{\mathbf{R}} -y_0(-\xi) e^{-|x-\xi|}\, d\xi \\ &= -\frac{1}{2}\int_{\mathbf{R}} e^{-|-x-\xi|} y_0(\xi)\, d\xi = -u_0(-x), \end{aligned}$$

which tells us that $u_0(x)$ is also an odd function.

On the other hand,

$$\begin{aligned} u_0'(0) &= -\frac{1}{2}\int_{-\infty}^{0} e^{\xi} y_0(\xi)\, d\xi + \frac{1}{2}\int_{0}^{-\infty} e^{-\xi} y_0(\xi)\, d\xi \\ &= \frac{1}{2}\int_{-\infty}^{0} e^{-\xi} y_0(\xi)\, d\xi + \frac{1}{2}\int_{0}^{-\infty} e^{-\xi} y_0(\xi)\, d\xi \\ &= \int_{0}^{-\infty} e^{-\xi} y_0(\xi)\, d\xi < 0. \end{aligned}$$

So this theorem follows from Lemma 2.3 directly. $\square$

Remark 3.3. Theorem 4.1 in [21] implies the CH2 system only blows up at the points $\rho = 0$. Similar result to MCH2 still remains open.

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